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A Two-Dimensional Fokker–Planck Equation Degenerating on a Straight Line

I. I. Fedchenia[1]

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By an example of a two-dimensional hydrodynamic system, second-order Langevin equations with two correlated noise sources are investigated. It is shown that the asymptotic expression ($t \to \infty$) for the stationary distribution function P depends on the order in which the limiting transitions $t \to \infty$ and $N_{22} \to 0$ (N_{22} is the power of one of the noises) are made. Using the method of local expansions in trigonometric form, approximate expressions are written for the distribution function P at small but finite N_{22} tending at $N_{22} \to 0$ to the known exact solution.

KEY WORDS: Noise in dynamic systems; bifurcations; Fokker–Planck equations; degenerating parabolic-type equations.

1. INTRODUCTION

Much work investigating the influence of noise perturbations on nonlinear systems has been devoted to the particular but practically important case of low noise, which permits the employment of various methods relying on the presence of a small parameter (the formulation of many problems and a review of the principal results are given in the classical monograph by Wentzel and Freidlin[(1)]). However, all this works consisders the case of uniformly low noise, i.e., all the diffusion tensor components in the Fokker–Planck equation are proportional to the small parameter $\varepsilon - \hat{D} = \varepsilon \hat{d}$. With a nonuniform tendency to zero of the diffusion tensor components (as well as in the case of degeneracy, $\det \hat{D} \to 0$), new effects are possible which are not observed in the uniform case. The following may be expected On diagonalization of the diffusion tensor the nonuniform tendency of its components to zero (or degeneration) will mean that some diagonal elements tend to zero faster than others (or, in the case of

[1] Institute of Physics, BSSR Academy of Sciences, 220602 Minsk, USSR.

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degeneration, they simply tend to zero while the remaining components do not tend to zero at all). This leads to the fact that in the phase space in the limit hyperplanes will arise transverse to which the noises do not act and the possibility of transition through such a plane will be determined by the noiseless side of the Langevin equations. But if the latter is such that the hyperplane arising in the limit becomes an unreachable boundary, the result of transition to the steady state must depend on the order of the limiting transitions: at small but finite elements of the nondegenerate diffusion tensor the system with $t \to \infty$ manages to spread throughout the phase space. In the opposite case, where the diagonal components of the diffusion tensor tend to zero prior to transition to the steady state, the system will always remain in the region bounded by the above-mentioned hyperplanes, which will be unreachable boundaries for this system.

The present paper considers the above questions by the example of a problem taken from the hydrodynamics of vortex flows in ellipsoidal containments[2] with regard to fluctuations. The equations for v_0, v_1, and v_2 of dimensionless lower modes of flow rates

$$\begin{aligned} \dot{v}_0 &= v_2^2 - v_1^2 - v_0 + R + f_0(t) \\ \dot{v}_1 &= v_0 v_1 - v_1 + f_1(t) \\ \dot{v}_2 &= -v_0 v_2 - v_2 + f_2(t) \end{aligned} \tag{I}$$

can be derived by applying the Galerkin procedure to the Helmholtz equation for the vortex motion of an ideal incompressible liquid inside an unequiaxal ellipsoid. Here R is the analog of the Reynolds number and f_0, f_1, and f_2 are fluctuation δ-correlated sources. Obukhov[2] has substantiated the application of such a single-parameter three-mode model. The field of application of the above simple model of fluctuation sources has been discussed by Klyatskin and Glukhovsky.[3]

The results presented in the sections that follow relate to a two-dimensional particular case of (I) where the component v_2 is not excited. In the new variables, which are more convenient for the calculation, Eqs. (I) take the form

$$\begin{aligned} \dot{x}_1 &= M - x_1 - x_2^2 + \eta_{x_1} \\ \dot{x}_2 &= x_1 \cdot x_2 + \eta_{x_2} \end{aligned} \tag{II}$$

where

$$M = R - 1, \qquad x_1 = v_0 - 1, \qquad x_2 = v_1, \qquad \langle \eta_{x_1}(t) \rangle = \langle \eta_{x_2}(t) \rangle = 0$$

$$\langle \eta_{\left\{ x_1 \atop x_2 \right\}}(t)\, \eta_{\left\{ x_1 \atop x_2 \right\}}(t') \rangle = N_{\left\{ 11 \atop 22 \right\}}\, \delta(t - t')$$

$$\langle \eta_{x_1}(t)\, \eta_{x_2}(t') \rangle = N_{12} \delta(t - t')$$

Such a particular case corresponds to Burger's model of the appearance of pulsations in a flow, where x_1 and x_2 relate respectively to the main flow and pulsations. Equations (II) are classified among the simple nonlinear quadratic systems in which bifurcations in the absence of noise have received much study (see, for example, ref. 4).

The present paper has resulted from an attempt to obtain an approximate stationary distribution function which at N_{22} (and arbitrary N_{11}) goes over into the exact solution obtained by Klyatskin and Glukhovsky[(3)] for the particular case of $N_{22}=0$. We give this solution in our notation, correcting an error in ref. 3 (where the modules sign is missing):

$$P = N|x_2|^{M/N_{11}-1} \exp[-(x_1^2 + +x_2^2)/2N_{11}] \tag{III}$$

In ref. 5 a stationary distribution function is formed that does not go into expression (III) at $N_{22} \to 0$. The question arises: to which situation does the solution (III) correspond and how one obtained an approximate expression for P that tends to (III) at $N_{22} \to 0$? The present paper gives the answers to these questions. It is shown that the stationary solution (III) can be obtained from the complete nonstationary solution with the following order of limiting transitions: $\lim_{t\to\infty} \lim_{N_{22}\to 0} P(t, x_1, x_2, N_{11}, N_{12}, N_{22})$, whereas the solution obtained in ref. 5 conforms to the limiting transition $\lim_{t\to\infty} P(t, x_1, x_2, N_{11}, N_{12}, N_{22})$, so that the attempt to derive (III) from it conforms to the reverse order of limiting transitions compared to the former case, $\lim_{N_{22}\to 0} \lim_{t\to\infty} P(t, x_1, x_2, N_{11}, N_{12}, N_{22})$. These two limits differ. This difference is due to the degeneration of the Fokker–Planck equation on the line $x_2 = 0$ at $N_{22} = 0$. In this case the phase space of the system of subdivided into two independent components in the sense that if at the initial instant of time the system is in one of these halves, then it will remain there at all subsequent instants. From this it follows, among other things, that the method used in ref. 5 is totally unsuited for obtaining an approximate expression for P tending to (III) at $N_{22} \to 0$, since it does not permit the boundary conditions to be specified, which, in the given case, must be chosen such that in the limit, P possesses a line of zeros $x_2 = 0$ which is present in expression (III).

Therefore, to answer the second question, we used the local method proposed in ref. 6, which enables one to specify the required boundary conditions. Approximate expressions for P have been obtained which tend to (III) at $N_{22} \to 0$ and are classified according to the solution of the corresponding problem of the first passage to the degeneration line (thus, they can be regarded as intermediate asymptotics for given times of the nonstationary distribution function). This situation is, in a sense, anologous to that which arises when approximate expressions are obtained for the passage time of the boundaries with all the noise powers uniformly tending to zero[7] when the determination of asymptotic expansion constants requires a knowledge of the stationary distribution function. In the present case, conversely, the intermediate asymptotics is classified according to the problem of the first passage time. As will be seen from the following, the majority of the results do not depend on particular details of the physical problem in question, a hydrodynamic type system. They are typical of two-dimensional systems with two or more steady equilibrium positions. Therefore, we first consider, in more detail as compared with ref. 6, the method of local expansions in polar coordinates and obtain general expressions for the coefficients of these epansions. Then the specifics of the problem will be taken into account.

2. THE METHOD OF LOCAL EXPANSIONS IN POLAR COORDINATES

Let us write the stationary Fokker–Planck equation

$$N_{11}\frac{\partial^2 P}{\partial x_1^2}+2N_{12}\frac{\partial^2 P}{\partial x_1\,\partial x_2}+N_{22}\frac{\partial^2 P}{\partial x_2^2}-\frac{\partial}{\partial x_1}K_1 P-\frac{\partial}{\partial x_2}K_2 P=0 \tag{1}$$

in the vector form (for convenience of comparison with the results of refs. 6 and 7, N_{ij} differs by 1/2 from the form used conventionally)

$$\begin{pmatrix}1\\1\end{pmatrix}^T\begin{pmatrix}\partial/\partial x_1 & 0\\ 0 & \partial/\partial x_2\end{pmatrix}\left[\begin{pmatrix}N_{11} & N_{12}\\ N_{12} & N_{22}\end{pmatrix}\begin{pmatrix}\partial/\partial x_1 & 0\\ 0 & \partial/\partial x_2\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}P-\begin{pmatrix}K_1(x)\\K_2(x)\end{pmatrix}P\right]=0 \tag{2}$$

On carrying out the change $x_1=r\cos\varphi$, $x_2=r\sin\varphi$, we have

$$\hat{D}=\begin{pmatrix}\dfrac{\partial}{\partial x_1} & 0\\ 0 & \dfrac{\partial}{\partial x_2}\end{pmatrix}=\begin{pmatrix}\cos\varphi\dfrac{\partial}{\partial r}-\sin\varphi\dfrac{1}{r}\dfrac{\partial}{\partial\varphi} & 0\\ 0 & \sin\varphi\dfrac{\partial}{\partial r}+\cos\varphi\dfrac{1}{r}\dfrac{\partial}{\partial\varphi}\end{pmatrix}$$

Let us seek P in a form that yields the exact solution (III),

$$P=N\phi(\varphi)r^k\exp[-U_1(\varphi)r-U_2(\varphi)r^2-\cdots-U_n(\varphi)r^n\cdots] \tag{3}$$

Substituting (3) into (2), we obtain the following chain of equalities for equal powers of r:

$$\begin{pmatrix}1\\1\end{pmatrix}^T \hat{D}\left\{r^{k-1}\begin{pmatrix}k\bar{N}_{11} & \bar{N}_{12}\\ k\bar{N}_{21} & \bar{N}_{22}\end{pmatrix}\begin{pmatrix}\phi\\ \partial\phi/\partial\varphi\end{pmatrix}\right\}=0 \quad (4)$$

$$\begin{pmatrix}1\\1\end{pmatrix}^T \hat{D}\left\{r^{k}\left[\begin{pmatrix}(k+1)\bar{N}_{11} & \bar{N}_{12}\\ (k+1)\bar{N}_{21} & \bar{N}_{22}\end{pmatrix}\begin{pmatrix}\phi U_1\\ \partial(\phi U_1)/\partial\varphi\end{pmatrix}+\phi\begin{pmatrix}K_1^0\\ K_2^0\end{pmatrix}\right]\right\}=0 \quad (5)$$

$$\begin{pmatrix}1\\1\end{pmatrix}^T \hat{D}\left\{r^{k+1}\left[\begin{pmatrix}(k+2)\bar{N}_{11} & \bar{N}_{12}\\ (k+2)\bar{N}_{21} & \bar{N}_{22}\end{pmatrix}\begin{pmatrix}\phi U_2\\ \partial(\phi U_2)/\partial\varphi\end{pmatrix}+\phi\begin{pmatrix}K_1^1\\ K_2^1\end{pmatrix}-\phi U_1\begin{pmatrix}K_1^0\\ K_2^0\end{pmatrix}\right.\right.$$
$$\left.\left.+\frac{U_1^2}{2}\begin{pmatrix}k\bar{N}_{11} & \bar{N}_{12}\\ k\bar{N}_{21} & \bar{N}_{22}\end{pmatrix}\begin{pmatrix}\phi\\ \partial\phi/\partial\varphi\end{pmatrix}-U_1\begin{pmatrix}(k+1)\bar{N}_{11} & \bar{N}_{12}\\ (k+1)\bar{N}_{21} & \bar{N}_{22}\end{pmatrix}\begin{pmatrix}\phi U_1\\ \partial(\phi U_1)/\partial\varphi\end{pmatrix}\right]\right\}=0$$

$$\vdots \quad (6)$$

$$\begin{pmatrix}1\\1\end{pmatrix}^T \hat{D}\left\{r^{k+n}\left[\begin{pmatrix}(k+n+1)\bar{N}_{21} & \bar{N}_{12}\\ (k+n+1)\bar{N}_{21} & \bar{N}_{22}\end{pmatrix}\begin{pmatrix}\phi U_{n+1}\\ \partial(\phi U_{n+1})/\partial\varphi\end{pmatrix}+\begin{pmatrix}F_1^{n+1}\\ F_2^{n+1}\end{pmatrix}\right]\right\}=0 \quad (7)$$

Here

$$\begin{pmatrix}\bar{N}_{11} & \bar{N}_{12}\\ \bar{N}_{21} & \bar{N}_{11}\end{pmatrix}=\begin{pmatrix}N_{11} & N_{12}\\ N_{12} & N_{22}\end{pmatrix}\begin{pmatrix}\cos\varphi & -\sin\varphi\\ \sin\varphi & \cos\varphi\end{pmatrix}$$
$$=\begin{pmatrix}N_{11}\cos\varphi+N_{12}\sin\varphi & -N_{11}\sin\varphi+N_{12}\cos\varphi\\ N_{12}\cos\varphi+N_{22}\sin\varphi & -N_{12}\sin\varphi+N_{22}\cos\varphi\end{pmatrix}$$

$$\begin{pmatrix}F_1^{n+1}\\ F_2^{n+1}\end{pmatrix}=-\frac{(\partial^{n+1}/\partial r^{n+1})(e^{-U})_0+(n+1)!\,U_{n+1}}{(n+1)!}$$
$$\times\begin{pmatrix}(k+n+1)\bar{N}_{11} & \bar{N}_{21}\\ (k+n+1)\bar{N}_{21} & \bar{N}_{22}\end{pmatrix}\begin{pmatrix}\phi\\ \partial\phi/\partial\varphi\end{pmatrix}$$
$$+\phi\sum_{i=0}^{n-1}\frac{1}{(n-i)!}\begin{pmatrix}K_1^i-\bar{N}_{12}(\partial U_{i+1}/\partial\varphi)\\ K_2^i+\bar{N}_{22}(\partial U_{i+1}/\partial\varphi)\end{pmatrix}$$
$$\times\frac{\partial^{n-i}}{\partial r^{n-i}}(e^{-U})_0+\phi\begin{pmatrix}K_1^n\\ K_2^n\end{pmatrix}$$

$$K_i^n=\frac{\partial^n}{\partial r^n}(K_i)_0;\qquad \frac{\partial^n}{\partial r^n}(e^{-U})_0=\frac{\partial^n}{\partial r^n}e^{-U}(r)\bigg|_{r=0} \quad (8)$$

It is seen that the nth equation of the above chain depends only on U_i with $i\leqslant n$, so that the whole chain can be solved sequentially.

Let us now perform in (4)–(7) the operation $\hat{D}$ in explicit form and equate to zero the expressions preceding r^n; we obtain

$$\left(\frac{N_{11}+N_{22}}{2}-f\right)\phi''+(k-1)\,f'\phi'+\left[\frac{N_{11}+N_{22}}{2}k^2+(k-2)\,kf\right]\phi=0 \qquad (9)$$

$$\left(\frac{N_{11}+N_{22}}{2}-f\right)(\phi U_1)''+k(\phi U_1)'+\left[\frac{N_{11}+N_{22}}{2}k^2+(k^2-1)f\right](\phi U_1)=F_1 \qquad (10)$$

$$\left(\frac{N_{11}+N_{22}}{2}-f\right)(\phi U_2)''+(k+1)(\phi U_2)' + \left[\frac{N_{11}+N_{22}}{2}(k+1)^2+k(k+2)f\right](\phi U_2)=F_2 \qquad (11)$$

$$\vdots$$

$$\left(\frac{N_{11}+N_{22}}{2}-f\right)(\phi U_n)''+(k+n-1)(\phi U_n)' + \left[\frac{N_{11}+N_{22}}{2}k+n-1)^2+(k+n-2)(k+n)f\right](\phi U_n)=F_n \qquad (12)$$

where

$$f=\frac{N_{11}-N_{22}}{2}\cos 2\varphi+N_{12}\sin 2\varphi$$

and

$$F_{n+1}=\begin{pmatrix}1\\1\end{pmatrix}^T\hat{D}\begin{pmatrix}F_1^{n+1}\\F_2^{n+1}\end{pmatrix}r^{-k-n+1}$$

In (8)–(12), let us make the substitutions

$$\phi=\bar{f}^{k/2}u_\phi \qquad (13)$$

and

$$\phi U_n=\bar{f}^{(k+n)/2}u_n \qquad (14)$$

where

$$\bar{f}=N_{11}\sin^2\varphi-2N_{12}\sin\varphi\cos\varphi+N_{22}\cos^2\varphi \qquad (15)$$

Then the equation for ϕ and the nth equation of the chain are reduced to the form

$$u''_\phi + \frac{\mathcal{J}'}{\mathcal{J}} + u'_\phi + \frac{k^2 \Delta^2}{\mathcal{J}^2} u_\phi = 0 \tag{16}$$

$$\vdots$$

$$u''_n + \frac{\mathcal{J}'}{\mathcal{J}} u'_n + \frac{(k+n)^2}{\mathcal{J}^2} \Delta^2 u_n = \mathcal{J}^{-(k+n)/2-1} F_n \tag{17}$$

where

$$\Delta^2 = N_{11} N_{22} = {}^2_{12} = N_{11} N_{22}(1-\rho^2)$$

ρ is the correlation coefficient, $-1 \leqslant \rho \leqslant 1$, and therefore $\Delta^2 \geqslant 0$.

It can readily be seen now that all u_n and u_ϕ are determined from the linear equations of one and the same form whose left-hand sides differ only in the number n. It is noteworthy that the homogeneous equation corresponding to (17) has an exact fundamental system of solutions

$$u_1 = \sin[(k+n)\,\Delta F], \qquad u_2 = \cos[(k+n)\,\Delta F]$$

where

$$F(\varphi) = \int \frac{d\varphi}{\mathcal{J}} = \frac{1}{\Delta} \left\{ \operatorname{arctg} \left[\left(\frac{N_{11}}{N_{22}} \right)^{1/2} \frac{\operatorname{tg} \varphi}{(1-\rho^2)^{1/2}} - \frac{\rho}{(1-\rho^2)^{1/2}} \right] \right.$$
$$\left. - \operatorname{arctg} \left[\frac{N_{11}}{N_{22}} \frac{1}{(1-\rho^2)^{1/2}} - \frac{\rho}{(1-\rho^2)^{1/2}} \right] \right\} \tag{18}$$

where the integration constant is chosen such that the limiting transitions

$$\lim_{N_{11} \to 0} F = \frac{1}{N_{22}} \operatorname{tg} \varphi, \qquad \lim_{N_{22} \to 0} F = -\frac{1}{N_{11}} \operatorname{ctg} \varphi$$

as well as some other transitions (see Appendix A) are realized so that all the equations of the chain are really solvable by quadratures. Moreover, as will be seen from the following, they are even solvable by elementary functions. It should be noted that this result is rather general, since the form of the equations in the chain and their solvability result only from the linearity of the Fokker–Planck equation and the two-dimensionality of the problem being considered. Considering, by induction, expressions (8) for F_n, one can easily see that they represent algebraic combinations $\{\sin[(k+n)\,\Delta F], \cos[(k+n)\,\Delta F]\}_{n=0}^{N}$ originating from the corresponding

U_n and $[\sin n\varphi, \cos n\varphi]_{n=0}^{N}$ originating from the corresponding $K_{1,2}^{n}$ for each fixed N, and making use of the identities

$$\sin \varphi = \tilde{f}^{1/2}(C_1^s u_1 + C_2^s u_2) \tag{19}$$

$$\cos \varphi = f^{1/2}(C_1^c u_1 + C_2^c u_2) \qquad (k+n=1) \tag{20}$$

$$\tilde{f} = [(C_1^s u_1 + C_2^s u_2)^2 + (C_1^c u_1 + C_2^c u_2)^2]^{-1} \tag{21}$$

(the form of the C_1^s, C_2^s, C_1^c, and C_2^c, which are functions of only N_{12}, N_{11}, and N_{22}, and the proof of (9)–(12) are given in Appendix A), we come to the conclusion that the solutions of all the equations of the chain are functions of $\Delta F(\varphi)$ only. The dependence of $\Delta F(\varphi)$ is given in Fig. 1. It is seen that any solution of a two-dimensional Fokker–Planck equation with boundary conditions specified for different intervals $(l\pi, \{l+1\}\pi)$, $l = 0, \pm 1, \pm 2, \ldots$, in the limit $N_{22} \to 0$ will be discontinuous. A continuous limiting solution of the type (III) can only be obtained by specifying the boundary conditions within open intervals $(l\pi, \{l+1\}\pi)$.

Physically, this corresponds to finding a system with probability 1 in any one interval being considered, i.e., if at the initial instant of time the system was in the sector $0 < \varphi < \pi$, then at all $t < t_1$, where t_1 is the first passage time to the line $\varphi = 0$, $\varphi = \pi$, the limit $\lim_{N_{22} \to 0} P(t)$ will be a continuous function of φ. Clearly $\lim_{t \to \infty} \lim_{N_{22} \to 0} P(t)$ will be the same.

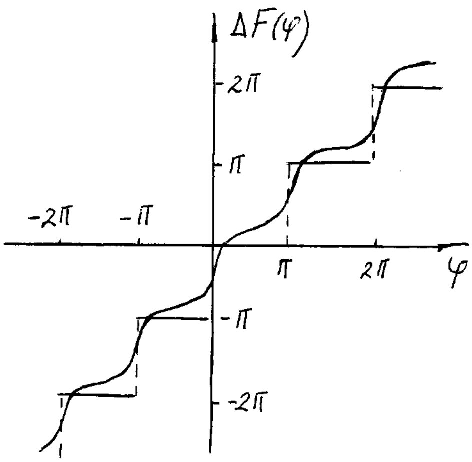

Fig. 1. F as a function of φ. The smooth curve corresponds to $N_{21} \neq 0$. The broken step curve is $\lim_{N_{22} \to 0} F(\varphi)$. At the points $\pm n\pi$, $n = 1, 2, \ldots$, $\lim_{N_{22} \to 0} F(\varphi) = F(0) \pm n\pi$.

3. CHECK OF LIMITING TRANSITIONS FOR *P*

Let us now proceed to the calculation of an approximate expression for P at small but finite N_{22}. We restrict ourselves, in the expansion of U, to the first two terms U_1 and U_2. In order to formulate the boundary conditions and uniquely define ϕ, U_1, and U_2, consider their limiting expressions, which are known from (III) for the limits of

$$\lim_{N_{22}\to 0} \phi = |\sin \varphi|^k \tag{22}$$

$$\lim_{N_{22}\to 0} U_1 = 0 \tag{23}$$

$$\lim_{N_{22}\to 0} U_2 = 1/2N_{11} \tag{24}$$

From the expression for ϕ and the form $\tilde{J}$ of (15) we find that $\lim_{N_{22}\to 0} u_\phi = \text{const}$, which can be assumed equal to unity. Thus,

$$\lim_{N_{22}\to 0} u_\phi = 1 \tag{25}$$

All the limiting expressions (13)–(25) have one feature in common: they are analytical functions of the angle φ for all values with zero derivatives of all orders. Choose $\varepsilon > 0$ and consider the boundary $\varphi = \varepsilon$, $\varphi = \pi - \varepsilon$. At times $t < t_1$, where t_1 is the first passage time of the boundary, at $N_{22} \ll 1$, P will be close to the limiting expression. Since the boundary $\varphi = 0$, $\varphi = \pi$ in unreachable at $N_{22} = 0$, the tending of ε to zero corresponds to the tending of t to infinity. Consequently,

$$\lim_{t\to\infty} \lim_{N_{22}\to 0} P = \lim_{\varepsilon\to 0} \lim_{N_{22}\to 0} P$$

The specifying of boundary conditions at $N_{22} \to 0$ at the boundary $\varphi = \varepsilon$, $\varphi = \pi - \varepsilon$ for P equal to the corresponding values of the limiting expression (III) is in agreement with the consideration of the problem at times $t < t_1$. As indicated above, the limiting expressions (23)–(25) are analytical in φ and have zero derivatives of all orders. To uniquely define the solutions of the equations of the chain (9)–(12), only two conditions are required for each equation, because they all are second-order equations. The question arises: derivatives of what order should be zero at the boundary $\varphi = \varepsilon$, $\varphi = \pi - \varepsilon$ as boundary conditions? Direct calculations show that the boundary conditions

$$u_\phi^{(N)}(\varepsilon) = u_\phi^{(N)}(\pi - \varepsilon) \tag{26}$$

$$U_n^{(N)}(\varepsilon) = U_n^{(N)}(\pi - \varepsilon) = 0 \qquad (n = 1, 2, \ldots) \tag{27}$$

give the solutions $u_{\phi N}$, U_{nN}, which have the same limits at $N_{22} \to 0$ for any N. It turns out that the order of derivatives in the boundary conditions of (26), (27) is associated with the expansion accuracy in the vicinity of $\varphi = 0$, $\varphi = \pi$. It is enough to show this only for $\varphi = 0$.

The Taylor formula for the N-fold differentiated functions is of the form[8]

$$U_n = U_n(0) + U_n'(0)\varphi + \cdots + \frac{1}{n!} U_n^{(N)}(\varepsilon)\varphi^N \tag{28}$$

where ε is related to φ through the relation $\varepsilon = \theta\varphi$, $(0 < \theta < 1)$. The equality $U_n^{(N)}(\varepsilon) = 0$ denotes that the remainder in (28) is discarded and the exact formula (8) is replaced by a φ-approximate formula.

Expressing φ in terms of ε and θ, substituting into (28), and discarding the remainder, we find that for given ε, and therefore for given t_1, U_n will be specified on the line $\varphi = 0$ by the approximate expression

$$U_n \approx U_n(0) + U_n'(0)\frac{\varepsilon}{\theta} + \cdots + \frac{1}{(n-1)!} U_n^{(N-1)}(0)\left(\frac{\varepsilon}{\theta}\right)^{N-1} \tag{29}$$

In further calculations we restrict ourselves to expression (29) in the first order in ε, i.e., we assume $N = 2$ in the boundary conditions (26), (27). Note that according to the formulation of boundary problems for fluctuating systems (see, for example, ref. 9), the boundary conditions at unreachable boundaries must be calculated rather than specified. The structure of the function U near the boundary exactly satisfies this requirement [at $\varepsilon = 0$ the boundary conditions are lost and only $U(0)$, which is calculated, remains].

The solution of the inhomogeneous equation (16) is of the form

$$u_\phi = \cos(k\,\Delta F) + C_\phi \sin(k\,\Delta F) \tag{30}$$

Here only one constant C_ϕ is left to be determined. This is because the second constant can be included in the normalization factor. Writing the boundary conditions of (26) for $N = 2$, we obtain the equation for determining C_ϕ:

$$u_\phi''(\varepsilon) = u_\phi''(\pi - \varepsilon) \tag{31}$$

Differentiating (30) twice with respect to φ, substituting into (31), and solving for C_ϕ, we obtain

$$C_\phi = \left\{ \frac{\mathcal{J}^1(\varepsilon)}{\mathcal{J}^2(\varepsilon)} \sin[k\,\Delta F(\varepsilon)] - \frac{\mathcal{J}^1(\pi-\varepsilon)}{\mathcal{J}^2(\pi-\varepsilon)} \sin[k\,\Delta F(\pi-\varepsilon)] + k\Delta \left[\frac{\cos[k\,\Delta F(\pi-\varepsilon)]}{\mathcal{J}^2(\pi-\varepsilon)} - \frac{\cos[k\,\Delta F(\varepsilon)]}{\mathcal{J}^2(\varepsilon)} \right] \right\}$$
$$\times \left\{ \frac{\mathcal{J}^1(\varepsilon)}{\mathcal{J}^2(\varepsilon)} \cos[k\,\Delta F(\varepsilon)] - \frac{\mathcal{J}^1(\pi-\varepsilon)}{\mathcal{J}^2(\pi-\varepsilon)} \cos[k\,\Delta F(\pi-\varepsilon)] + k\Delta \left[\frac{\sin[k\,\Delta F(\varepsilon)]}{\mathcal{J}^2(\varepsilon)} - \frac{\sin[k\,\Delta F(\pi-\varepsilon)]}{\mathcal{J}^2(\pi-\varepsilon)} \right] \right\}^{-1} \quad (32)$$

Using expression (15) for $\mathcal{J}$,

$$\mathcal{J}(\varepsilon) = N_{11}\sin^2\varepsilon - 2N_{12}\sin\varepsilon\cos\varepsilon + N_{22}\cos^2\varepsilon \quad (33)$$

$$\mathcal{J}(\pi-\varepsilon) = N_{11}\sin^2\varepsilon + 2N_{12}\sin\varepsilon\cos\varepsilon + N_{22}\cos^2\varepsilon \quad (34)$$

and for $\mathcal{J}^1$,

$$\mathcal{J}^1(\varepsilon) = (N_{11} - N_{22})\sin 2\varepsilon - 2N_{12}\cos 2\varepsilon \quad (35)$$

$$\mathcal{J}^1(\pi-\varepsilon) = -(N_{11} - N_{22})\sin 2\varepsilon - 2N_{12}\cos 2\varepsilon \quad (36)$$

as well as expanding $\sin(k\Delta F)$ and $\cos(k\Delta F)$ into series in $\Delta \sim N_{22}^{1/2}$, we obtain

$$C_\phi \sim \Delta k[F(\varepsilon) - F(\pi-\varepsilon)]$$

and taking into account the approximate expressions for $F(\varepsilon)$, $F(\pi-\varepsilon)$,

$$F(\varepsilon) = (-N_{11}\operatorname{tg}\varepsilon + N_{12})^{-1}, \qquad F(\pi-\varepsilon) = (N_{11}\operatorname{tg}\varepsilon + N_{12})^{-1}$$

obtained by expanding $\operatorname{arctg} x$ and using expression (18) for $F(x)$ at large x, and their sum and difference,

$$F(\varepsilon) + F(\pi-\varepsilon) = \frac{2N_{12}}{N_{12}^2 - N_{11}^2\operatorname{tg}^2\varepsilon}, \qquad F(\varepsilon) - F(\pi-\varepsilon) = -\frac{2N_{11}\operatorname{tg}\varepsilon}{N_{12}^2 - N_{11}^2\operatorname{tg}^2\varepsilon} \quad (37)$$

we see that C_ϕ tends to zero at fixed ε as $N_{22}^{1/2}$. Hence $\lim_{N_{22}\to 0} u_\phi = 1$, which coincides with the required limiting value (25).

Now proceed to the solution of Eq. (10) for U_1 [or, respectively, (17) for U_1]. Substituting into the expression for F_1 both $K_1^0 = M$ and $K_2^0 = 0$ for the problem at hand, we obtain

$$F_1 = M(k\phi\cos\varphi - \phi'\sin\varphi) \quad (38)$$

Replacing $\cos\varphi$ and $\sin\varphi$ by their expressions in terms of $\sin(\Delta F)$ and $\cos(\Delta F)$ according to (19) and (20), we obtain the right-hand side of Eq. (17) for u_1,

$$\mathcal{J}^{-(k+3)/2}F_1=\frac{Mk}{2\mathcal{J}^2}\{C_c\cos[(k-1)\,\Delta F]+C_s\sin[(k-1)\,\Delta F]\} \qquad (39)$$

Here C_c and C_s are functions of only N_{11}, N_{12}, and N_{22} and they are of the order of $N_{22}^{1/2}$. Their expressions as well as the proof of equality to zero of the coefficients of $\cos[(k+1)\,\Delta F]$ and $\sin[(k+1)\,\Delta F]$ in (39) are given in Appendix A. It can easily be seen that a particular solution of the inhomogeneous equation (17) with the right-hand side of (39) for u_1 is the equation

$$u_{1p_1}=\frac{M}{8\Delta^2}\{C_c\cos[(k-1)\,\Delta F]+C_s\sin[(k-1)\,\Delta F]\} \qquad (40)$$

It is more convenient, however, to use as a particular solution the expression

$$u_{1p_2}=\frac{M}{4\Delta^2}\left(C_c\sin(k\,\Delta F)\sin(\Delta F)+\frac{1}{2}C_s\left\{\sin[(k-1)\,\Delta F]-\frac{k-1}{k+1}\sin[(k+1)\,\Delta F]\right\}\right) \qquad (41)$$

which has the order $N_{22}^{1/2}$, while u_{1p_1} in (40) has the order $N_{22}^{-1/2}$, and therefore tends to infinity with N_{22} tending to zero. This expression is derived by subtracting from (40) the solutions of the homogeneous equation $\sin[(k+1)\,\Delta F]$ and $\cos[(k+1)\,\Delta F]$ with the corresponding coefficients. Then the general solution of (17) for $n=1$ will be of the form

$$u_1=C_1^1\cos[(k+1)\,\Delta F]+C_2^1\sin[(k+1)\,\Delta F]+u_{1p_2} \qquad (42)$$

Using expressions (13) and (14) relating u_1 and U_1, and ϕ and u_ϕ, we obtain the expression for U_1:

$$U_1=\mathcal{J}^{1/2}u_1/u_\phi \qquad (43)$$

We obtain from the boundary condition of (27) a system of linear algebraic equations with respect to C_1^1 and C_2^1:

$$\begin{aligned}A_1(\varepsilon)C_1^1+B_1(\varepsilon)C_2^1&=D_1(\varepsilon)\\A_1(\pi-\varepsilon)C_1^1+B_1(\pi-\varepsilon)C_2^1&=D_1(\pi-\varepsilon)\end{aligned} \qquad (44)$$

where

$$A_1(\varphi)=\left[\left(\frac{\mathcal{J}^{1/2}(\varphi)}{u_\phi(\varphi)}\right)'' - \frac{(k+1)^2\Delta^2}{\mathcal{J}^{3/2}(\varphi)u_\phi}\right]\cos[(k+1)\,\Delta F(\varphi)$$
$$-\frac{(k+1)\Delta}{\mathcal{J}^{1/2}(\varphi)}\left(\frac{1}{u_\phi(\varphi)}\right)'\sin[(k+1)\,\Delta F(\varphi)] \tag{45}$$

$$B_1(\varphi)=\left[\left(\frac{\mathcal{J}^{1/2}(\varphi)}{u_\phi(\varphi)}\right)'' - \frac{(k+1)^2\,\Delta^2}{\mathcal{J}^{3/2}(\varphi)\,u_\phi(\varphi)}\right]\sin[(k+1)\,\Delta F(\varphi)]$$
$$+\frac{(k+1)\Delta}{\mathcal{J}^{1/2}(\varphi)}\left(\frac{1}{u_\phi(\varphi)}\right)'\cos[(k+1)\,\Delta F(\varphi)] \tag{46}$$

$$D_1(\varphi)=-\left(\frac{u_{2p_2}(\varphi)\,\mathcal{J}^{1/2}(\varphi)}{u_\phi(\varphi)}\right)'' \tag{47}$$

One can easily evaluate the behavior of A_1, B_1, and D_1 at $\Delta\sim N_{22}^{1/2}\to 0$:

$$A_1(\varphi)\sim -N_{11}^{1/2}\sin\varphi\cos[(k+1)\,\Delta F(\varphi)]\sim \text{const} \tag{48}$$

$$B_1(\varphi)\sim -N_{11}^{1/2}\sin\varphi\sin[(k+1)\,\Delta F(\varphi)]$$
$$\sim \Delta(k+1)\,F(\varphi)\sin\varphi\, N_{11}^{1/2}\sim N_{22}^{1/2} \tag{49}$$

$$\det\begin{pmatrix} A_1(\varepsilon) & B_1(\varepsilon)\\ A_1(\pi-\varepsilon) & B_1(\pi-\varepsilon)\end{pmatrix}$$
$$\sim N_{11}\sin\varepsilon\sin\{(k+1)\,\Delta[F(\varepsilon)-F(\pi-\varepsilon)]\}$$
$$\sim 2\Delta\,\frac{(k+1)N_{11}^2\sin\varepsilon\,\mathrm{tg}\,\varepsilon}{N_{11}^2\,\mathrm{tg}^2\,\varepsilon-N_{12}^2}\sim N_{22}^{1/2} \tag{50}$$

$$D_1(\varphi)\sim N_{22}^{1/2} \tag{51}$$

by virtue of the special choice of the particular solution (41);

$$\det\begin{pmatrix} D_1(\varepsilon) & B_1(\varepsilon)\\ D_1(\pi-\varepsilon) & B_1(\pi-\varepsilon)\end{pmatrix}\sim N_{22} \tag{52}$$

by virtue of the above estimates (49), (51); and

$$\det\begin{pmatrix} A_1(\varepsilon) & D_1(\varepsilon)\\ A_1(\pi-\varepsilon) & D_1(\pi-\varepsilon)\end{pmatrix}\sim N_{22}^{1/2} \tag{53}$$

by virtue of the estimates (48), (51). Then from (50) and (52) if follows that

$$C_1^1 = \det\begin{pmatrix} D_1(\varepsilon) & B_1(\varepsilon) \\ D_1(\pi-\varepsilon) & B_1(\pi-\varepsilon)\end{pmatrix} \Big/ \det\begin{pmatrix} A_1(\varepsilon) & B_1(\varepsilon) \\ A_1(\pi-\varepsilon) & B_1(\pi-\varepsilon)\end{pmatrix} \sim N_{22}^{1/2} \quad (54)$$

and from (50), (53)

$$C_2^1 = \det\begin{pmatrix} A_1(\varepsilon) & D_1(\varepsilon) \\ A_1(\pi-\varepsilon) & D_1(\pi-\varepsilon)\end{pmatrix} \Big/ \det\begin{pmatrix} A_1(\varepsilon) & B_1(\varepsilon) \\ A_1(\pi-\varepsilon) & B_1(\pi-\varepsilon)\end{pmatrix} \sim O(1) \quad (55)$$

If we now take into account that $\sin[(k+1)\,\Delta F] \sim (k+1)\,\Delta F \sim N_{22}^{1/2}$, then $U_1 \sim N_{22}^{1/2}$ and therefore tends to zero at $N_{22} \to 0$, which just provides the required limit of (23).

Equation (11) in U_2 is sollved exacty in the same manner. First calculate F_2 by substituting into the corresponding equation (8) $K_1^1 = -\cos\varphi$ and $K_2^1 = 0$ for the problem at hand. In addition, neglect the terms $\sim U_1 \sim N_{22}$, i.e., consider only the dominant asymptotic term plus corrections $\sim N_{22}^{1/2}$. Neglect also the terms $\sim M^2$, which corresponds to the consideration of solutions with maxima close to the origin of coordinates, since only for these is the solution in the form of (3) with the first two U_1 and U_2 other than zero a good approximation (in principle, this does not impose any restrictions on the proposed procedure, but makes the calculations less awkward). So, for this case

$$F_2 \cong \phi + (k\phi\cos\varphi - \phi'\sin\varphi)\cos\varphi \quad (56)$$

Using expression (38) for F_1 written as a function of ΔF and substituting into (56) ϕ written according to (13) and (30) and $\cos\varphi$ written according to (20), we obtain the right-hand side of Eq. (17) for u_2

$$\mathcal{J}^{-(k+2)/2}F_2 \cong \frac{1+kC_0/4}{\mathcal{J}^2}\cos(k\Delta F) + \frac{C_\phi + kS_0/4}{\mathcal{J}^2}\sin(k\,\Delta F) + \frac{kC^-}{4\mathcal{J}^2}\cos[(k-2)\,\Delta F] + \frac{kS^-}{4\mathcal{J}^2}\sin[(k-2)\,\Delta F] \quad (57)$$

Here C_0, S_0, C^-, and S^- are only functions of N_{11}, N_{12}, and N_{22} which are given in Appendix A; C_ϕ is function in expression (32) for ϕ.

As in the case of u_1, one can easily find the apparent particular solution:

$$u_{2p_1} = \frac{\cos(k\,\Delta F)}{4(k+1)\Delta^2} + \frac{C_0 k}{16(k+1)\Delta^2}\cos(k\,\Delta F) + \frac{C_\phi + S_0 k/4}{4(k+1)\Delta^2}\sin(k\,\Delta F) + \frac{C^-}{32\Delta^2}\cos[(k-2)\,\Delta F] + \frac{S^-}{32\Delta^2}\sin[(k-2)\,\Delta F] \quad (58)$$

which has the order of N_{22}^{-1} at $N_{22} \to 0$ and is therefore unsuitable for the calculation of the limiting transition. The particular solution

$$u_{2p_2} = \frac{1}{2N_{11}^2}\cos[(k+2)\,\Delta F] + \frac{\sin[(k+1)\,\Delta F]\sin(\Delta F)}{2(k+1)\Delta^2} + \frac{C_0 k}{8(k+1)\Delta^2}\sin[(k+1)\,\Delta F]\sin(\Delta F) + \frac{C^-}{16\Delta^2}\sin(k\,\Delta F)\sin(2\,\Delta F) + \frac{C_\phi + S_0 k/4}{4(k+1)\Delta^2}\left\{\sin(k\,\Delta F) - \frac{k}{k+2}\sin[(k+2)\,\Delta F]\right\} + \frac{S^-}{32\Delta^2}\left\{\sin[(k-2)\,\Delta F] - \frac{k-2}{k+2}\sin[(k+2)\,\Delta F]\right\} \tag{59}$$

is more convenient. This solution is obtained from (58) by adding, with the corresponding coefficients, the solution of the homogeneous equation $\sin[(k+2)\,\Delta F]$ and $\cos[(k+2)\,\Delta F]$. As can readily be seen, the first two terms tend to $1/2N_{11}^2 \sin^2\varphi$, whereas all the other terms, as follows from the corresponding formulas of Appendix A, tend to zero as $N_{22}^{1/2}$ does. The complete solution will have the form

$$u_2 = C_1^2\cos[(k+2)\,\Delta F] + C_2^2\sin[(k+2)\,\Delta F] + u_{2p_2} \tag{60}$$

Using expressions (13) and (14) relating u_2 and U_2, and ϕ and u_ϕ, we obtain

$$U_2 = \mathcal{J}u_2/u_\phi \tag{61}$$

From the boundary condition of (28) we obtain a system of linear algebraic equations with respect to C_1^2 and C_2^2:

$$\begin{aligned} A_2(\varepsilon)C_1^2 + B_2(\varepsilon)C_2^2 &= D_2(\varepsilon) \\ A_2(\pi-\varepsilon)C_1^2 + B_2(\pi-\varepsilon)C_2^2 &= D_2(\pi-\varepsilon) \end{aligned} \tag{62}$$

where

$$A_2(\varphi) = \left[\left(\frac{\mathcal{J}(\varphi)}{u_\phi(\varphi)}\right)'' - \frac{(k+2)^2\Delta^2}{\mathcal{J}(\varphi)\,u_\phi(\varphi)}\right]\cos[(k+2)\,\Delta F(\varphi)] - (k+2)\Delta\left[\frac{\mathcal{J}'(\varphi)}{\mathcal{J}(\varphi)\,u_\phi(\varphi)} + 2\left(\frac{1}{u_\phi(\varphi)}\right)'\right]\sin[(k+2)\,\Delta F(\varphi)] \tag{63}$$

$$B_2(\varphi) = \left[\left(\frac{\mathcal{J}(\varphi)}{u_\phi(\varphi)}\right)'' - \frac{(k+2)^2\Delta^2}{\mathcal{J}(\varphi)\,u_\phi(\varphi)}\right]\sin[(k+2)\,\Delta F(\varphi)] + (k+2)\Delta\left[\frac{\mathcal{J}'(\varphi)}{\mathcal{J}(\varphi)\,u_\phi(\varphi)} + 2\left(\frac{1}{u_\phi(\varphi)}\right)'\right]\cos[(k+2)\,\Delta F(\varphi)] \tag{64}$$

$$D_2(\varphi) = -\left(\frac{u_{2p_2}\mathcal{J}}{u_\phi}\right)'' \tag{65}$$

It can easily be seen that the estimates for A_2, B_2, and

$$\det\begin{pmatrix} A_2(\varepsilon) & B_2(\varepsilon) \\ A_2(\pi-\varepsilon) & B_2(\pi-\varepsilon)\end{pmatrix}$$

coincide with the corresponding estimates for (48)–(50).

To show that $\lim_{N_{22}\to 0} C_1^2 = 0$, it is sufficient to establish that $D_2(\varphi)\sim N_{22}^{1/2}$. For this purpose, in turn, it is sufficient to show that

$$\lim_{N_{22}\to 0}\left(\frac{u_{2p_2}\mathcal{J}}{u_\phi}\right)=0$$

since $u_{2p_2}\mathcal{J}/u_\phi$ is analytical in $N_{22}^{1/2}$. Since all the terms of the particular solution (59), except for the first two, are $\sim N_{22}^{1/2}$, we have

$$\begin{aligned}
\lim_{N_{22}\to 0}\left(\frac{u_{2p_2}\mathcal{J}}{u_\phi}\right)'' &= \lim_{N_{22}\to 0}\left(\frac{\mathcal{J}(\varphi)}{u_\phi(\varphi)}\left\{\frac{1}{2N_{11}^2}\cos[(k+2)\,\Delta F(\varphi)]\right.\right.\\
&\quad\left.\left.+\frac{\sin[(k+1)\,\Delta F(\varphi)]\sin[\Delta F(\varphi)]}{2(k+1)\Delta^2}\right\}\right)''\\
&= \lim_{N_{22}\to 0}\left[\left(\frac{\mathcal{J}(\varphi)}{u_\phi(\varphi)}\right)''\left(\frac{\cos[(k+2)\,\Delta F(\varphi)]}{2N_{11}^2}\right.\right.\\
&\quad\left.+\frac{\sin[(k+1)\,\Delta F(\varphi)]\sin[\Delta F(\varphi)]}{2(k+1)\Delta^2}\right.\\
&\quad+\frac{1}{\mathcal{J}(\varphi)}\left(\frac{\mathcal{J}(\varphi)}{u_\phi(\varphi)}\right)'\left(\frac{\cos[(k+1)\,\Delta F(\varphi)]\sin[\Delta F(\varphi)]}{\Delta}\right.\\
&\quad\left.+\frac{\sin[(k+1)\,\Delta F(\varphi)]\cos[\Delta F(\varphi)]}{\Delta}\right)\\
&\quad+\frac{\mathcal{J}(\varphi)}{u_\phi(\varphi)}\left(\frac{\cos[(k+1)\,\Delta F(\varphi)]\cos[\Delta F(\varphi)]}{\mathcal{J}(\varphi)}\right.\\
&\quad-\frac{\mathcal{J}'(\varphi)\cos[(k+1)\,\Delta F(\varphi)]\sin[\Delta F(\varphi)]}{2\Delta\mathcal{J}^2(\varphi)}\\
&\quad\left.\left.-\frac{\mathcal{J}'(\varphi)\sin[(k+1)\,\Delta F(\varphi)]\cos[\Delta F(\varphi)]}{2\Delta\mathcal{J}^2(\varphi)(k+1)}\right)\right]\\
&= \lim_{N_{22}\to 0}\frac{1}{N_{11}}\left[(\cos^2\varphi-\sin^2\varphi)\left(1+\frac{\cos^2\varphi}{\sin^2\varphi}\right)\right.\\
&\quad\left.-4\frac{\cos^2\varphi}{\sin^2\varphi}+3\frac{\mathrm{co}^2\varphi}{\sin^2\varphi}+1\right]=0
\end{aligned}$$

So, $D_2(\varphi) \sim N_{22}^{1/2}$. Hence, as is the case with (54) and (55), we obtain for C_1^2 and C_2^2

$$C_1^2 = \det\begin{pmatrix} D_2(\varepsilon) & B_2(\varepsilon) \\ D_2(\pi-\varepsilon) & B_2(\pi-\varepsilon)\end{pmatrix} \Big/ \det\begin{pmatrix} A_2(\varepsilon) & B_2(\varepsilon) \\ A_2(\pi-\varepsilon) & B_2(\pi-\varepsilon)\end{pmatrix} \sim N_{22}^{1/2}$$
$$C_2^2 = \det\begin{pmatrix} A_2(\varepsilon) & D_2(\varepsilon) \\ A_2(\pi-\varepsilon) & D_2(\pi-\varepsilon)\end{pmatrix} \Big/ \det\begin{pmatrix} A_2(\varepsilon) & B_2(\varepsilon) \\ A_2(\pi-\varepsilon) & B_2(\pi-\varepsilon)\end{pmatrix} \sim O(1) \tag{66}$$

If now we take into account that the first two terms of (59) tend to $1/2N_{11}^2 \sin^2\varphi$ and all the other terms in the solution (60) tend to zero, as indicated above, we obtain, for (61) at $N_{22} \to 0$, the required limit (4), $1/2N_{11}$.

It follows from the form of the exact limiting solution (III) that

$$\lim_{N_{22} \to 0} U_n = 0 \qquad (n \geqslant 3) \tag{67}$$

Using the method of mathematical induction, we show that U_n obtained from Eqs. (12) actually has such a limit. Expression (18) for $F(\varphi)$ and the structure F_n in expressions (12) enable us to state that all U_n are analytical functions of $N_{22}^{1/2}$. Consequently, for U_n to be of the order of $N_{22}^{1/2}$ it is necessary and sufficient that the limit (67) be met. Also, if we manage to show that $F_n \sim N_{22}^{1/2}$, then, using a technique similar to that used in obtaining the partial solution u_{1p_2} in (41) and u_{2p_2} in (59), we can form the partial solutions $u_{np2} \sim N_{22}^{1/2}$ and exactly show, as in the case of C_1^1 in (54) and C_1^2 in (66), that $C_1^n \sim N_{22}^{1/2}$.

So, taking into account that $U_1 \sim N_{22}^{1/2}$, we obtain for the dominant asymptotic term of F_3

$$F_3 \sim \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}^T \hat{D}\phi \begin{pmatrix} -\sin^2\varphi - U_2 M \\ \sin\varphi\cos\varphi \end{pmatrix} \right\} r^{-k-1}$$
$$= M\left[\frac{\partial(U_2\phi)}{\partial\phi}\sin\varphi - (k+2)\cos\varphi(U_2\phi)\right] + \sin\varphi\frac{\partial\phi}{\partial\varphi} + \cos\varphi\,\phi$$

whence

$$\lim_{N_{22} \to 0} F_3 = (k + 1 - M/N_{11})\cos\varphi\sin^k\varphi \tag{68}$$

Obviously, for the limit (68) to be equal to zero at any φ it is necessary that $k = M/N_{11} - 1$; or, comparing with the exact limiting solution (III), we note that k coincides with the limiting value. We can make sure, however, that it is impossible to eliminate the terms $\sim N_{22}^{1/2}$

in the expression for F_3 with the aid of k, which depends only on the parameters of the problem, but not on φ. So, the obtained value of k is the best that can be attained approximating the sought solution by expression (3).

The next step of reasoning on induction should be made separately for even and odd n. In (8) let $n=2l-1$. Then from formula (B.4) of Appendix B and from the assumption that all $U_n \sim N_{22}^{1/2}$ $(1 \leqslant n \leqslant 2l-1,\ n \neq 2)$ it follows that the dominant asymptotic term in (8) $\sim O(1)$ is of the form

$$\begin{pmatrix} F_1^{2l} \\ F_2^{2l} \end{pmatrix} \sim -\begin{pmatrix} N_{11}\left[\dfrac{\cos\ \varphi(k+2l)}{2l!}\left(\dfrac{\partial^{2l}e^{-U}}{\partial r^{2l}}\right)_0 \phi - \dfrac{\sin\varphi}{(2l)!}\phi'\left(\dfrac{\partial^{2l}e^{-U}}{\partial r^{2l}}\right)_0 \right. \\ \left. + \dfrac{\phi\cos\varphi}{(2l-2)!}\left(\dfrac{\partial^{2l-2}e^{-U}}{\partial r^{2l-2}}\right)_0\right] \\ 0 \end{pmatrix} \tag{69}$$

Equation (69) also takes into account that for the given problem $K_1^1 = -\cos\varphi$, $K_2^1 = 0$, and K_1^n, $K_2^n = 0$ $(n \geqslant 3)$ and the form of $\bar{N}_{ij}$. Because only terms consisting of products of U_2 can be of $O(1)$, we obtain from formula (A4) of Appendix B for dominant asymptotic terms

$$\left(\frac{\partial^{2l}e^{-U}}{\partial r^{2l}}\right)_0 \sim C_{2l-1}^1 C_{2l-3}^1 \cdots C_3^1(-2U_2)^{2l} \tag{70}$$

$$\left(\frac{\partial^{2l-2}}{\partial r^{2l-2}}e^{-U}\right)_0 \sim C_{2n-3}^1 \cdots C_3^1(-2U_2)^{2l} \tag{71}$$

Substituting (70) and (71) into (69), we obtain, taking into account that $C_{2l-1}^1 = 2l-1$,

$$\lim_{N_{22}\to 0} F_1^{2l} = \lim_{N_{22}\to 0}\left\{\frac{r^{k+2l+1}}{(2l-2)!}\left[\frac{N_{11}}{2l-1}C_{2n-1}^1(-2U_2)^{2l} + (2U_2)^{2l-1}\right]\sin^k\varphi\cos\varphi \right.$$

$$\left. \times [C_{2l-3}^1 \cdots C_3^1]\right\}$$

$$= \frac{r^{k+2l-1}}{(2n-2)!}[C_{2l-3}^1 \cdots C_3^1]\left(-\frac{1}{N_{22}}\right)^{2n-1}\sin^k\varphi\cos\varphi$$

$$\times\left[\frac{N_{11}(2n-1)}{2n-1}\left(-\frac{2}{2n_{11}}\right)+1\right] = 0$$

Let now $n=2l$. Then the dominant asymptotic term for $F_{2l+1} \sim O(1)$ has the form

$$F_{2l+1} \sim \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}^T \hat{D}\phi \left[\begin{pmatrix} K_0^1 \\ K_0^2 \end{pmatrix} \frac{1}{(2l)!} \frac{\partial^{2l}}{\partial r^{2l}} (e^{-U})_0 + \begin{pmatrix} K_2^1 \\ K_2^2 \end{pmatrix} \frac{1}{(2l-2)!} \frac{\partial^{2l-2}}{\partial r^{2l-2}} (e^{-U})_0 \right] r^{k+2l} \right\} r^{-k-2l+1} \tag{72}$$

Using expressions (70) and (71) and substituting into (72) the expression $K_1^0 = M$, $K_2^0 = 0$, $K_1^2 = -\sin^2\varphi$, and $K_2^2 = \sin\varphi\cos\varphi$ for the given problem, we obtain

$$\begin{aligned} F_{2l+1} &\sim \frac{C_{2n-3}^1 \cdots C_3^1}{(2n-2)!} (-2U_2)^{2n-1} \left[\cos\varphi\, M\phi(-2U_2) + \cos\varphi\,\phi + \sin\varphi \frac{d\phi}{d\varphi} \right] \\ &\sim \frac{C_{2n-3}^1 \cdots C_3^1}{(2n-2)!} (-2U_2)^{2n-1} \cos\varphi \sin^k\varphi \left(k+1-\frac{M}{N_{11}} \right) \end{aligned} \tag{73}$$

Hence

$$\lim_{N_{22}\to 0} F_{2l+1} = 0 \qquad \text{if} \quad k = M/N_{11} - 1$$

i.e., it coincides with the value obtained when considering the case $n=2$.

So, we have proved that the expression (3) for P in which ϕ and U_n are obtained under boundary conditions (26) and (27) really tends to the limiting exact solution (III) with all additions to the limiting exact expressions for ϕ and $U_n \sim N_{22}^{1/2}$. The last result comes from the fact that the correlation coefficient ρ is nonzero. With $\rho=0$ it can easily be checked, that all additions have the order of N_{22}.

4. EVALUATION OF APPROXIMATE SOLUTION ERROR

Let us now evaluate the error which we make by leaving a finite number of terms U_n ($n=1,\ldots, N$) in expression (3) for P. If we substitute the difference $\Delta P = P - P_{\text{approx}}$ between the exact solution P and the approximation (3) into the initial equation (2), then, due to the linearity of the Fokker–Planck equation,

$$L_{\text{FP}}\, \Delta P = -L_{\text{FP}} P_{\text{approx}} \tag{74}$$

where L_{FP} is the linear differential operator of Eq. (2). Expanding the analytical part P_{approx} into a Taylor series at zero with the remainder in the Lagrange form and substituting into (74), we obtain

$$L_{\text{FP}}\, \Delta P = -\frac{1}{(N+1)!} L_{\text{FP}} \left[\frac{\partial^{N+1}}{\partial r^{N+1}} P_{\text{approx}}(\xi)\, r^{k+N+1} \right] \tag{75}$$

where $0 < \xi < r$. Due to the fact that the first N terms of the expansion in (75) are equal to zero under the conditions of finding the functions U_n ($n = 1,..., N$), it follows from the above estimates of U_n and their structure that in the interval ($\varepsilon < \varphi < \pi - \varepsilon$) all U_n ($n = 1,..., N$) are bounded and they all have bounded derivatives with respect to φ. Then, taking into account that L_{FP} causes the power of r to decrease by 2, we may write the estimate uniform in the interval stated

$$|L_{\mathrm{FP}}\, \Delta P| \leqslant \frac{r^{k+N-1}(k+N+1)(k+N)}{(N+1)!}\, \delta(r)\, C_1 \tag{76}$$

the power of $\delta(r)$ being not lower than the power of K_1 and K_2. Taking into account the analyticity of P_{approx} in $N_{22}^{1/2}$ and the tendency $P_{\mathrm{approx}} \to_{N_{22} \to 0} P$, we may write

$$|L_{\mathrm{FP}}\, \Delta P| \leqslant N_{22}^{1/2} r^{k+N-1} \frac{(k+N+1)(k+N)}{(N+1)!}\, \delta(r)\, C_2 \tag{77}$$

It is seen from (76) and (77) that the local error [$r < R$, where R is the radius of the convergence circle of the series $\sum U_n r_n$; the convergence itself follows from Kovalevskaya's theorem, the conditions of which are satisfied by Eq. (1) with boundary conditions (26), (27)] decreases at both $N_{22} \to 0$ and $N \to \infty$, i.e., with increasing number of calculated U_n. Setting the first derivative with respect to r from expression (3) with two calculated U_1 and U_2 equal to zero, we obtain the equation for the maximum curve:

$$r_{\mathrm{max}} = -\frac{U_1(\varphi)}{4U_2(\varphi)} + \left[\frac{U_1^2(\varphi)}{16U_2^2(\varphi)} + \frac{k}{2U_2(\varphi)}\right]^{1/2} \tag{78}$$

If we refer to the estimate of (77), we may say that the distribution function maximum is well approximated at small M, N_{22}, and k [see (43) for U_1], i.e., in the cases where the maximum lies in the vicinity of the expansion point $r = 0$, which agrees with the locality of the expansions used in the present paper.

5. CONCLUSION

The results obtained in the present paper may be of interest for systems with more than one position of equilibrium. In this case, at noises smaller in the direction between these two states and at times shorter than the first time of leaving the region of most probable stay, the distribution function will have the form similar to that obtained by specifying the boundary conditions of the type (26), (27). And if we take into account that the time of observing the system is always finite, the obtained intermediate asymptotic distribution function describes the situation more realistically than the true stationary distribution function does.

APPENDIX A

In this Appendix, relations (19)–(21) are derived and the coefficients C_c and C_s entering into expression (39) and C_0, S_0, C^-, and S^- entering into expression (57) are calculated as well as their equality to zero at $\sin[(k+1)\,\Delta F]$ and $[(k+1)\,\Delta F]$ in expression (39).

Relations (19)–(21) result from observing the fact that Eq. (12) at $k=1$, $n=1$, and a zero right-hand side is the equation of harmonic vibrations

$$\mathcal{J}\tilde{U}''+\mathcal{J}\tilde{U}=0 \tag{A.1}$$

where $\tilde{U}=\phi U_1$ ($k=0$). By replacing (14) at $k=0$, $\phi U=\mathcal{J}^{1/2}u$, it is reduced to an equation of the type of (17),

$$U''+\frac{\mathcal{J}'}{\mathcal{J}}u'+\frac{\Delta^2}{\mathcal{J}^2}u=0 \tag{A.2}$$

which has fundamental solutions $\Delta^{-1}\sin(\Delta F)$, $\cos(\Delta F)$. By replacing (14), the solution of (A.1) may be expressed as a linear combinations of these solutions, namely, $\sin\varphi$ and $\cos\varphi$. For $\sin\varphi$ we have

$$\begin{aligned}\tilde{U}&=\sin\varphi=\mathcal{J}^{1/2}u, \qquad u=\sin\varphi/\mathcal{J}^{1/2}\\ \tilde{U}'&=\cos\varphi=\mathcal{J}'u/2\mathcal{J}^{1/2}+\mathcal{J}^{1/2}u'\\ u'&=\cos\varphi-\mathcal{J}'\sin\varphi/2\mathcal{J}\\ &=(N_{22}\cos\varphi-N_{12}\sin\varphi)/\mathcal{J}\end{aligned} \tag{A.3}$$

Any solution of (A.2) is expressed as a linear combination of its fundamental solutions

$$\begin{aligned}u&=C_1^s\Delta^{-1}\sin\Delta F+C_2^s\cos\Delta F\\ u'&=(C_1^s\cos\Delta F-\Delta C_2^s\sin\Delta F)/\mathcal{J}\end{aligned} \tag{A.4}$$

Equations (A.3) and (A.4) must hold at any $\varphi=\alpha$. Then

$$\begin{aligned}&C_1^s\Delta^{-1}\sin[\Delta F(\alpha)]+C_2^s\cos[\Delta F(\alpha)]=\sin\alpha/\mathcal{J}^{1/2}(\alpha)\\ &C_1^s\cos[\Delta F(\alpha)]-C_2^s\Delta\sin[\Delta F(\alpha)]=(N_{22}\cos\alpha-N_{12}\sin\alpha)/\mathcal{J}^{1/2}(\alpha)\\ &\det\begin{vmatrix}\Delta^{-1}\sin[\Delta F(\alpha)] & \cos[\Delta F(\alpha)]\\ \cos[\Delta F(\alpha)] & -\Delta\sin[\Delta F(\alpha)]\end{vmatrix}=-1\end{aligned} \tag{A.5}$$

from which we obtain

$$C_1^s = \{\Delta \sin\alpha \sin[\Delta F(\alpha)] + (N_{22}\cos\alpha - N_{12}\sin\alpha)\cos[\Delta F(\alpha)]\}/\mathcal{T}^{1/2}(\alpha)$$
$$C_2^s = \{\Delta \sin\alpha \cos[\Delta F(\alpha)] - (N_{22}\cos\alpha - N_{12}\sin\alpha)\sin[\Delta F(\alpha)]\}/\Delta\mathcal{T}^{1/2}(\alpha) \tag{A.6}$$

For further transformations of C_1^s and C_2^s we write the explicit form $\Delta F(\alpha)$ accoding to formula (18):

$$\Delta F(\alpha) = \phi(\alpha) - \phi_0 \tag{A.7}$$

where

$$\phi(\alpha) = \operatorname{arctg}\left[\left(\frac{N_{11}}{N_{22}}\right)^{1/2}\frac{\operatorname{tg}\alpha}{(1-\rho^2)^{1/2}} - \frac{\rho}{(1-\rho^2)^{1/2}}\right] \tag{A.8}$$

$$\phi_0 = \operatorname{arctg}\left[\left(\frac{N_{11}}{N_{22}}\right)^{1/2+\varepsilon}\frac{1}{(1-\rho^2)^{1/2}} - \frac{\rho}{(1-\rho^2)^{1/2}}\right] \tag{A.9}$$

ε is chosen later on the basis of the conditions of realizing the necessary limiting transitions.

Substituting (A.7)–(A.9) into (A.6) and taking into account the formulas

$$\sin(\operatorname{arctg} A) = A(1+A^2)^{-1/2}, \qquad \cos(\operatorname{arctg} A\} = (1+A^2)^{-1/2}$$

we obtain

$$C_1^s = (N_{22}^{1+\varepsilon} - N_{11}^{\varepsilon} N_{12})[N_{11}^{1+2\varepsilon} - 2\rho(N_{11}N_{22})^{1/2+\varepsilon} + N_{22}^{1+2\varepsilon}]^{-1/2}$$
$$C_2^s = N_{11}^{\varepsilon}[N_{11}^{1+2\varepsilon} - 2\rho(N_{11}N_{22})^{1/2+\varepsilon} + N_{22}^{1+2\varepsilon}]^{-1/2} \tag{A.10}$$

As expected, the relations (A.10) are independent of α.

Now check that

$$\lim_{N_{22}\to 0} \tilde{U} = \sin\varphi$$

That is, relations (A.3) also hold in the limiting case. Then

$$\lim_{N_{22}\to 0} \tilde{U} = N_{11}^{1/2}\sin\varphi(-C_1^s N_{11}^{-1}\operatorname{ctg}\varphi + C_2^s) = \sin\varphi$$

Absolutely analogous calculations for $\tilde{U} = \cos\varphi$ yield $\cos\varphi = \mathcal{T}^{1/2}u$, where

$$u = C_1^c\,\Delta^{-1}\sin(\Delta F) + C_2^c\cos(\Delta F)$$
$$C_1^c N_{11}^{1/2}(N_2^{1/2+\varepsilon}\rho - N_{11}^{1/2+\varepsilon})[N_{11}^{1+2\varepsilon} - 2\rho(N_{11}N_{22})^{1/2+\varepsilon} + N_{22}^{1+2\varepsilon}]^{-1/2}$$
$$C_2^c = N_{22}^{\varepsilon}[N_{11}^{1+2\varepsilon} - 2\rho(N_{11}N_{22})^{1/2+\varepsilon} + N_{22}^{1+2\varepsilon}]^{-1/2}$$

Requiring that the transition $\lim_{N_{22}\to 0} \tilde{U} = \cos\varphi$ be realized, we obtain

$$\lim_{N_{22}\to 0} \tilde{U} = N_{11}^{1/2} \sin\varphi(-C_1^c N_{11}^{-1} \operatorname{ctg}\varphi + C_2^c)$$

whence it is seen that the second term in the parentheses must be equal to zero, which is possible only in the case where ε is any number strictly greater than zero. For convenience, we henceforth take $\varepsilon = 1/2$. Let us show how (39) is obtained from (38) and calculate C_c and C_s entering into (39).

Substituting the expression for ϕ, according to (30), into (38), we obtain

$$\begin{aligned} F_1 &= M(k\phi\cos\varphi - \phi'\sin\varphi) \\ &= Mk\bar{J}^{k/2-1}\{[N_{22}\cos\varphi - (N_{12} + \Delta C_\phi)\sin\varphi]\cos(k\,\Delta F) \\ &\quad + [C_\phi N_{22}\cos\varphi - (C_\phi N_{12} - \Delta)\sin\varphi]\sin(k\,\Delta F)\} \end{aligned} \tag{A.11}$$

Substituting into (A.10) sin φ and co φ expressed as $\sin(\Delta F)$ and $\cos(\Delta F)$ according to formulas (19) and (20) and rewriting, using identities of elementary trigonometry, the products of trigonometric functions in terms of functions of the sum and difference, we obtain

$$\begin{aligned} F_1 &= 2^{-1}Mk\bar{J}^{(k-1)/2}[R_c\cos[(k+1)\,\Delta F] + R_s\sin[(k+1)\,\Delta F] \\ &\quad + C_c\cos[(k-1)\,\Delta F] + C_s\sin[(k-1)\,\Delta F] \end{aligned} \tag{A.12}$$

where

$$\begin{aligned} R_c &= (C_2^c N_{22} - C_2^s N_{12} - C_1^s) + C_\phi^s(N_{12}\Delta^{-1}C_1^s - N_{22}\Delta^{-1}C_1^c - \Delta C_2^s) \\ &= \{(N_{22}^{3/2} - N_{11}^{1/2}N_{12} - N_{22}^{3/2} + N_{22}^{1/2}N_{11}\rho) \\ &\quad + C_\phi\Delta^{-1}[N_{12}N_{22}^{1/2}(N_{22} - N_{11}\rho) - N_{22}N_{11}^{1/2}(N_{22}\rho - N_{11}) - \Delta^2 N_{11}^{1/2}]\} \\ &\quad \times (N_{11}^2 - 2\rho N_{11}N_{22} + N_{22}^2)^{-1/2} \equiv 0 \\ R_s &= (C_1^c\Delta^{-1}N_{22} - C_1^s\Delta^{-1}N_{12} + \Delta C_2^s) + C_\phi(C_2^c N_{22} - C_2^s N_{12} - C_1^s) \equiv 0 \\ C_c &= (N_{22}C_2^c - N_{12}C_2^s + C_1^s) + C_\phi\Delta^{-1}(N_{22}C_1^c - N_{12}C_1^s - \Delta^2 C_2^s) \\ &= 2\,\frac{N_{22}^{1/2}[N_{22} - N_{11}\rho - C_\phi N_{11}(1-\rho^2)^{1/2}]}{(N_{11}^2 - 2\rho N_{11}N_{22} + N_{22}^2)^{1/2}} \\ C_s &= \frac{2N_{22}^{1/2}[N_{22} - N_{11}\rho + C_\phi N_{11}(1-\rho^2)^{1/2}]}{(N_{11}^2 - 2\rho N_{11}N_{22} + N_{22}^2)^{1/2}} \end{aligned}$$

Similarly, (57) is obtained from (56). In this case the coefficients C_0, S_0, C^-, and S^- have the form

$$C^- = C_c C_2^c + C_s C_1^c \Delta^{-1}; \qquad S^- = C_s C_2^c - C_c C_1^c \Delta^{-1}$$

APPENDIX B

In this Appendix the general expression for $d^n \exp[\varphi(x)]/dx^n$ is derived which is used in calculating the limits and estimating errors in Section 3. This formula is also useful in calculating the series convergence radius in nonlinear approximations of the type of formula (3), because it has an explicit dependence on n. Hereafter the designation $d^n y/dx^n = y^{(n)}$ is used.

Using the Leibnitz rule,[8] we can write

$$(e^{\varphi(x)})^{(n)} = (e^{\varphi}\varphi^{(1)})^{(n-1)} = \sum_{k=0}^{n-1} C_{n-1}^{k}(e^{\varphi})^{(n-k-1)}\varphi^{(k+1)} \tag{B.1}$$

Let us do this once again:

$$(e^{\varphi})^{(n-k-1)} = (e^{\varphi}\varphi^{(1)})^{(n-k-2)} = \sum_{k'=0}^{n-k-2} C_{n-k-2}^{k'}(e^{\varphi})^{(n-k'-k-2)}\varphi^{(k'+1)} \tag{B.2}$$

Substituting (B.2) into (B.1), we obtain

$$(e^{\varphi})^{(n)} = \sum_{k=0}^{n-1}\sum_{k'=0}^{n-k-2} C_{n-1}^{k}C_{n-k-2}^{k'}(e^{\varphi})^{n-k'-k-2}\varphi^{(k+1)}\varphi^{(k'+1)} \tag{B.3}$$

It is seen that in (B.3) the derivative indices are always $\leqslant n-2$. Proceeding in this way until a derivative of zero index is obtained and making the necessary rearrangement of the terms, we obtain the final formula:

$$\begin{aligned}(e^{\varphi})^{(n)} = e^{\varphi}\Bigg[&\varphi^{(n)} + \sum_{k=0}^{n-2} C_{n-1}^{k}\varphi^{(n-k-1)}\varphi^{(k+1)} \\ &+ \sum_{k=0}^{n-2}\sum_{k'=0}^{n-k-3} C_{n-1}^{k}C_{n-k-2}^{k'}\varphi^{(n-k-k'-2)}\varphi^{(k'+1)}\varphi^{(k+1)} \\ &+ \sum_{k=0}^{n-2}\sum_{k'=0}^{n-k-3}\sum_{k''=0}^{n-k-k'-4} C_{n-1}^{k}C_{n-k-2}^{k'}C_{n-k-k'-3}^{k''}\varphi^{(n-k-k'-k''-3)} \\ &\times\varphi^{(k'+1)}\varphi^{(k+1)}\varphi^{(k''+1)} \\ &+\cdots+\sum_{k=0}^{n-2}\sum_{k'=0}^{n-3}\cdots\sum_{k^{(n)}=0}^{n-n} C_{n-1}^{0}C_{n-2}^{0}\cdots C_{1}^{0}(\varphi^{(1)})^{n}\Bigg]\end{aligned}$$

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